Biomechanics of Running and Walking

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Abstract

Running and walking are integral to most sports and there is a considerable amount of mathematics involved in examining the forces produced by each foot contacting the ground. In this paper we discuss biomechanical terms related to running and walking. We then use experimental ground reaction force data to calculate the impulse of running, speed-walking, and walking. We then mathematically model the vertical ground reaction force curves for both running and walking, successfully reproducing experimental data. Finally, we discuss the biological implications of the mathematical models and give suggestions for numerous classroom or research projects.

Introduction

Running speed is essential to most sports, whether it is the ability to beat a defender, run faster than an opponent or develop enough take-off velocity to achieve distance or height on a jump. Running tends to occur at faster speeds than walking, although speed walkers can achieve speeds of up to 4.6 meters per second using an unusual gait in which the hip is dropped each step. Running is defined as a gait in which there is an aerial phase, a time when no limbs are touching the ground. Aside from wind resistance and gravity, there are no external forces applied to the body during this aerial phase. Therefore, it is the stance phase (the time when a limb is in contact with the ground) of running that must be modified in order to change speed.

We can measure the forces involved in the stance phase of running. This is called kinetics, which is the study of movement and the forces involved in producing it. Running forces are usually measured using a force plate. A force plate takes advantage of Newton's third law of motion: for every action there is an equal and opposite reaction. When we step on the ground we produce a vector of force that is generally downward and backward. The ground produces a force that is generally upward and forward, and it is this ground reaction force (GRF) that is measured by the force plate.

In order to run faster, stance time must be shorter. Shortening stance time, however, gives less time to produce an impulse, so the peak forces must be higher. Impulse is calculated

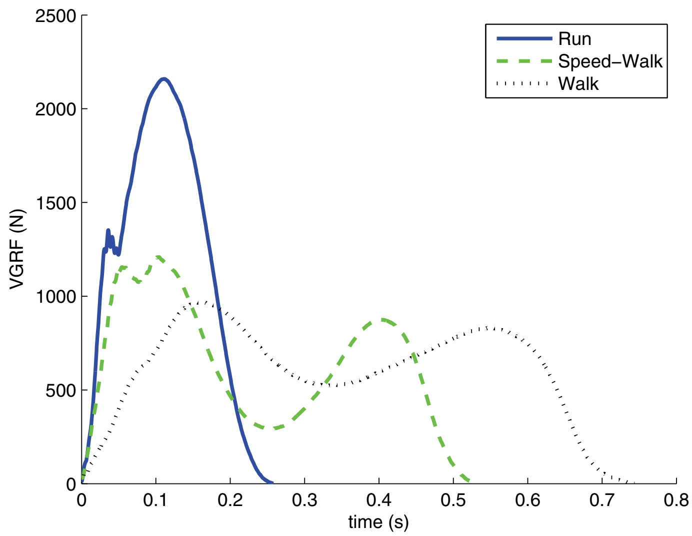

Figure 1: The curves correspond to VGRF experimental data for one step of a run (trial 8 from the spreadsheet), speed-walk (trial 1), and walk (trial 1).

by computing the impact force multiplied by the time over which the impact force acts. The impulse for one step of a run is approximately constant regardless of the method of running. Consider the force on the knee joint for a stiff-legged running step and a compliant bent-knee step; recall that the force is the impulse divided by the impact time. For a stiff-legged run, the impact time is short, so the force on the knee is large. However, with a compliant bent-knee after impact, the impact time is longer, so the force on the knee is smaller (so bending their knees helps runners keep joint forces lower!). It is these high peak ground reaction forces at impact that contribute to the transmission of shock through the skeletal system and have been associated with overuse injuries such as shin splints and stress fractures [1, 2]. Mathematically, the impulse, I, of a step for running or walking is given by $I = \int \mathrm{F} dt$, where F is the force.

Consider a three-dimensional coordinate system with orthogonal directions x, y, and z. Imagine walking along the x-axis, with the y-axis pointing toward the sky. Now think about the forces that the ground imparts on your every step as equal and opposite reactions; these are the forces measured by a force plate. The force with the largest magnitude that the ground imparts on your body is the vertical ground reaction force (VGRF), which is in the y-direction. The antero-posterior (fore-aft) force, in the x-direction, is approximately a factor of 10 less than the VGRF. The medial-lateral force, in the z-direction, is approximately a factor of 100 less than the VGRF in both running and walking. Due to its importance, VGRF data will be the focus of this article. Data from one step of a run, walk, and speed-walk are shown in Figure 1.

It may be helpful to further examine the shape of the VGRF data in Figure 1 for each gait. The walking VGRF data exhibits two noticeable peaks. The first peak corresponds to the period just after the heel touches the force plate and the center of gravity is traveling down toward the ground, resulting in an increased reaction force from the ground in the vertical direction. The second peak corresponds to the toe pushing off of the force plate, applying a force into the ground which is matched by an increase in the ground reaction force. The dip in the middle of these peaks occurs when the center of gravity is rising away from the ground, thus decreasing the force of the body on the ground and therefore decreasing the reaction force the ground is exerting on the body in the vertical direction. Two peaks are also observed in the speed-walking data, but there is an additional bump in the first peak called the impact peak. If you look back at the VGRF data for walking, you can also see a slight impact peak. The impact peak is the force associated with the foot striking the ground prior to the center of gravity traveling downward, resulting in the larger VGRF peak. Finally, the running VGRF data has a single peak with a very sizable impact peak; notice in running the center of gravity is traveling down toward the ground on impact and only changes directions once to move away. As the center of gravity travels toward the plate, the body is increasing the force it is applying on the ground and therefore the ground is applying a greater force on the body.

Educationally, impulse is another useful application of integration. Since impulse is defined as the area under the curve of VGRF, $I = \int \text{VGRF}dt$, the data from the force plate can be used to illustrate the importance of either integration in general or specifically numerical integration. Also, recall that the VGRF is a force; so using Newton's second law, $F = ma$ or $VGRF = ma$, we can find the acceleration to be $a = \frac{VGRF}{m}$. Now, in order to calculate the velocity, v, integrate the acceleration to obtain $v = \int \frac{\text{VGRF}}{m}dt$.

Applications

An excel spreadsheet with data from walking, speed-walking, and running ground reaction forces is located at http://www.math.jmu.edu/~tongen/TongenWunderlich.xls. These data were collected on a Kistler 9286AA force plate in the JMU Animal Movement laboratory. The subject weighed 77.51 kg (760.4 N), and the sampling rate of the force plate was 2500 Hz, which means that the force plate calculated the forces every $\frac{1}{2500} = .0004$ seconds. In the spreadsheet, Fz is the VGRF, Fy is the antero-posterior force, and Fx is the medial-lateral force; all data have units of Newtons.

In the following, we use the spreadsheet data to calculate the impulse for each type of gait. We then use the data to model the VGRF curves for running while learning some biomechanical terminology. Finally, we use the data to model the VGRF curves for walking.

After each discussion, we list possible directions for classroom projects or research projects.

Numerically Calculating Impulse

Impulse describes the force applied over a period of time. Recall that the impulse, I, is given by $I = \int \text{VGRF}dt$. Looking at the data in Figure 1, which gait has the largest impulse? Does the height of the running curve overcome the width of the walking gait?

In order to calculate the impulse, we need to approximate the area under the VGRF curve using Riemann sums. Recall from Calculus that Riemann sums are (usually introduced as) a way of breaking a continuous curve into little tiny rectangles and later trapezoids. Through Riemann sums, we calculate an approximation to the area under the curve. Many students cannot imagine a reason that one would ever want to make this type of approximation. However, experimental data is a case where we don't have an actual function describing the relationship between the independent and dependent variables. Therefore, calculating the impulse of the VGRF curve is a great motivating example for numerical integration.

In this paper, we will use the trapezoidal rule. Therefore, $I = \int \text{VGRF}dt \approx \sum_{k=1}^{n} \frac{F_k + F_{k+1}}{2} \cdot \Delta t$, where F_k is the force measurement from the force plate at time t_k and $\Delta t = t_k - t_{k-1}$, which is constant throughout the trial. Using the above formula, we calculate the average impulse of the experimental data for running, speed-walking, and walking to be 309, 336, and 438 N·s, respectively.

While it is difficult to compare these values because we are only measuring one limb, and during walking/speed-walking the load of the body is shared with another limb, we can begin to examine the effects of speed using these data. We achieve a similar impulse in very different ways in running and speed walking. Running is characterized by high peak forces and short contact times, while walking is characterized by lower peak forces and longer contact times. At higher speeds, contact times will be even shorter, but this will necessitate higher peak forces in order to support body weight.

Run with it

This section, and subsequent "Run with it" sections, will give ideas for future classroom or research projects. Some of the projects will be accessible to students in courses as early as Calculus, while others will require more advanced mathematical training.

1. Given a VGRF curve, find the velocity curve and interpret the results.

2. Given a VGRF curve, plot the position of the center of mass and interpret the results.

3. Verify the impulse-momentum relationship from the data.

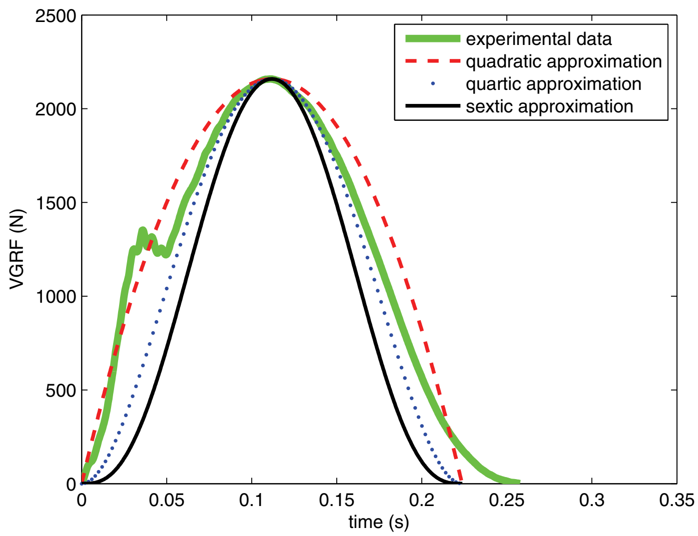

Figure 2: The thick curve corresponds to VGRF experimental data for one step of a run. The other curves are polynomial approximations to the VGRF curves with T=0.2237 s and M=2158.9 N.

Running Model

With a numerical approximation of the impulse of running in hand, we can now model the shape of the VGRF curve of a single step of a run. We initially assume that the VGRF curve is a parabola given by $F_R(t) = -at^2 + bt + c$, where $0 \leq t \leq T$ and T is the amount of time that the foot is in contact with the ground. The initial point of contact occurs at $t = 0$ while the last point of contact occurs at $t = T$. Notice that the first term of $F_R(t)$ is negative; this sign occurs because the VGRF curve is shaped like a parabola opening downward. Given $F_R(t)$, the impulse is given by $I = \frac{-aT^3}{3} + \frac{bT^2}{2} + cT$.

There are numerous ways to determine the constants a, b, and c. Two logical boundary conditions are $F_R(0) = 0$ and $F_R(T) = 0$. These boundary conditions reduce the original equation to $F_R(t) = -a(t^2 - Tt) = -at(t - T)$ with only one unknown parameter a. Now we can use $F_R(t)$ to calculate the impulse, $I = \int_0^T F_R(t)dt = \frac{aT^3}{6}$. Therefore, the size of the impulse depends on the size of the yet unknown parameter a.

Visually, the most straightforward final assumption to find the final unknown parameter, a, is that the maximum value occurs approximately at the midpoint of the VGRF curve, i.e. $F_R'(\frac{T}{2}) = 0$. Interestingly, this condition is automatically satisfied due to the choice of a parabola to model the VGRF curve and the boundary conditions that have already been used.

Since the maximum occurring at the midpoint is automatically satisfied, we can examine other possibilities to determine the final unknown, a. Assume the impulse, say I^*, is given or calculated from the experimental data. Then, we can determine the quadratic curve for VGRF in running as $F_R^{(1)}(t) = -\frac{6I^*}{T^3}t(t-T)$, where I^* is the known impulse. We have called this approximation $F_R^{(1)}(t)$ to differentiate it from the following approximation.

Another possiblity is to assume that the maximum VGRF value, M, is given, i.e. $F_R(T/2) = M$. Then, $F_R^{(2)}(t) = -\frac{4M}{T^2}t(t-T)$ with an impulse given by

$$I = \frac{2MT}{3}. \tag{1}$$

The size of the impulse depends on what happens to the value of MT, where M is the maximum value of the VGRF and T is the amount of time that the runner's foot is in contact with the ground. Notice that the mathematics completely agrees with the definition in the Introduction.

We can now use (1) to calculate the impulse of $F_R^{(2)}(t)$ from Figure 2. With $T = 0.2237$ seconds and $M = 2158.8$ N, the impulse is $I = 321.964$ N·s, which is larger than the numerical value of 309 N·s. This discrepancy, along with the poor visual match of the quadratic approximation with the experimental data, motivates the following section where we introduce additional biomechanical concepts.

Inverse problem

Although the above section is a nice mathematical exercise, notice in Figure 2 that the quadratic approximation does a very poor job of approximating the VGRF curve for running. Before we increase the order of the approximating polynomial, we examine another biomechanical concept before returning to the mathematics.

Body weight, BW, can be approximated from the VGRF curve. A person running has to support an average of one body weight during the step cycle. Therefore, the BW is approximated by the line through the VGRF curve such that the area above the line but below the curve is equal to the area below the line and above the curve. Mathematically, the above concept for calculating BW is called the average value of the function. Recall that the average value is defined as

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx \tag{2}$$

and calculates the average value for a function over some interval.

We can calculate BW mathematically using the above definition to get $\int_0^{t_1}(BW - F_R(t))dt + \int_{t_2}^{T}(BW - F_R(t))dt = \int_{t_1}^{t^2}(F_R(t) - BW)dt$, where t_1 is the first intersection and t_2 is the second intersection of the line BW and $F_R(t)$. We can simplify the above expression to get $BW \cdot t_1 - \int_0^{t_1} F_R(t)dt + BW(T - t_2) - \int_{t_2}^{T} F_R(t)dt = \int_{t_1}^{t^2} F_R(t)dt - BW(t_2 - t_1)$. Collecting terms simplifies the equation to $BW \cdot T = \int_0^T F_R(t)dt$. Finally, solving for body weight yields

$BW = \frac{\int_0^T F_R(t)dt}{T} = \frac{I}{T}$. By substituting $a = 0$ and $b = T$ into (2), it is straightforward to see that the calculation of BW is what we call the average value in mathematics. Also, there is an immediate connection between body weight and impulse.

Now we can apply this concept of average value of a function to the two quadratic approximations from the previous section. Given that we know the VGRF curve, where the maximum value M is easily calculated, using (1) we are able to find the body weight of an individual as $BW = \frac{2M}{3}$. The other way to think about this equation is that knowing the BW allows us to approximate the maximum of the VGRF as $M = \frac{3}{2}BW$. However, examining our data shows that the maximum VGRF values for a run are approximately 3 times BW, not $1\frac{1}{2}$ BW. Therefore, not only does the curve do a poor job of approximating the VGRF curve as seen in Figure 2, the biomechanics are also not captured with the quadratic approximation.

Now, return to the idea that the impulse I^* is known. As we saw earlier, one way to calculate impulse is to approximate it numerically from the experimental data. Recall that you can approximate $I^* = BW \cdot T$. So, impulse is approximated by body weight multiplied by the amount of time the foot is in contact with the ground. With this assumption, the maximum value in the case where we knew the impulse, I^*, can be determined. We need to calculate $F_R^{(1)}(\frac{T}{2})$. After some algebra, we find $F_R^{(1)}(\frac{T}{2}) = \frac{3}{2}BW$. This result means that $F_R^{(1)} = F_R^{(2)}$ and both quadratic approximations to the VGRF curve in the previous section for running give the exact same answer, which is not a very good answer.

Assuming a quartic approximation to the VGRF curve with boundary conditions given by $F_R(0) = 0, F_R(T) = 0, F_R'(0) = 0, F_R'(T) = 0$, and $F_R(T/2) = M$ is given by $F_R(t) = \frac{-16M}{T^4}t^2(t-T)^2$ and is the dotted curve in Figure 2. This result implies that the maximum for the quartic approximation occurs at $F_R(\frac{T}{2}) = \frac{15}{8} \cdot BW$. Similarly, a sextic approximation to the VGRF curve with boundary conditions given by $F_R(0) = 0, F_R(T) = 0, F_R'(0) = 0, F_R'(T) = 0, F_R''(0) = 0, F_R''(T) = 0$, and $F_R(T/2) = M$ is given by $F_R(t) = \frac{64M}{T^6}t^3(t-T)^3$ and is the thin solid curve in Figure 2. The maximum value occurs at $F_R(\frac{T}{2}) = \frac{35}{16} \cdot BW$.

Although the sextic polynomial approximation is closer to predicting the maximum height of the curve based on the BW, visually it appears that the quartic approximation is the best. As we increase the order of the polynomial and continue to set higher derivatives at the boundary to zero, the polynomials will look more and more like a delta function and less and less like the experimental data. These results certainly motivate numerous projects for the "Run with it" section!

Run with it

Below are a few suggested classroom or research projects for students.

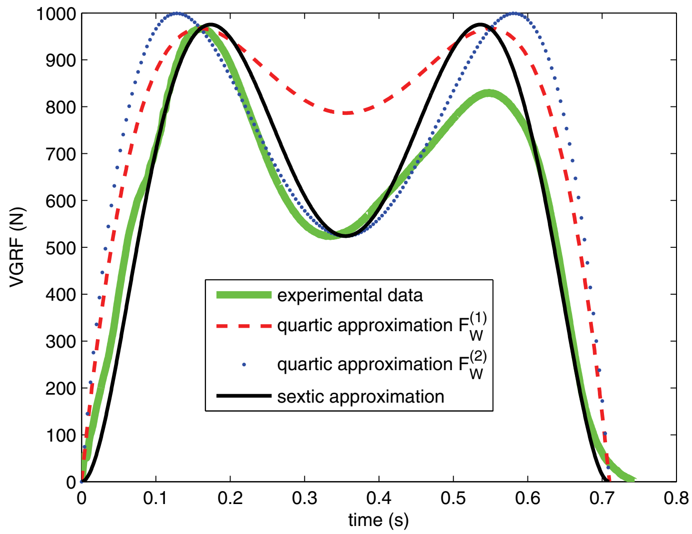

Figure 3: The thick curve corresponds to VGRF experimental data for one step of a walk. The other curves are polynomial approximations to the VGRF curves with T=0.7099 s, M=966.86 N, t_{max}=0.16 s, and m=524 N.

1. Are there other, more physically reasonable, ways to approximate the VGRF curve for running?

2. For the method and boundary conditions used in the above modeling, what is the limit of the maximum value as the order of approximation goes to infinity? Does the limit increase without bound or is it going to a limiting value?

3. In the above mathematical modeling, we have used polynomial interpolation, i.e. trying to model the data with a polynomial. Is it possible, and beneficial, to use Fourier series or another set of basis functions to model the data?

Walking Model

Figure 3 shows a typical VGRF curve for one step of a walk. Notice that this curve cannot be modeled by a quadratic function. However, a quartic function is the simplest first approximation. We will assume that the VGRF curve for walking is approximated by $F_W(t) = -at^4 + bt^3 + ct^2 + dt + f$, where $0 \leq t \leq T$. Modeling the VGRF curves for walking is much more difficult than running, because of two additional unknown parameters. Below, we will provide two examples of solving for the five unknown parameters; however, we will leave it as an exercise to the reader to derive their own walking FUNctions!

Using the same boundary conditions that were used with running, we assume $F_W(0) =$

$F_W(T) = 0$; this assumption simplifies the approximation to $F_W(t) = -at(t^3 - T^3) + b(t^2 - T^2) + ct(t-T) = -t(t-T)(at^2 - bt + aTt - c - Tb + aT^2)$. Next, assuming that $F'_W(\frac{T}{2}) = 0$, i.e. there is a local minimum in the middle, gives $F_W(t) = -at(t^3 - T^3) + 2aTt(t^2 - T^2) + ct(t-T) = -at(t^3 - 2Tt^2 + T^3) + ct(t-T) = -t(t-T)(at^2 - aTt - c - aT^2)$. These assumptions mean two unknown parameters remain.

If the location of the maxima in the VGRF curve data is known, this information will allow the determination of the final two unknown parameters for the quartic function that best models the data. Assume that symmetric maxima occur at t_{max} and $T - t_{max}$, i.e. $F'_W(t_{max}) = F'_W(T - t_{max}) = 0$. Astonishingly, these two conditions yield only one unknown; similarly to the case with running, these two conditions are duplicates and only determine one of the remaining unknown parameters. We are left with $F_W(t) = -at(t^3 - 2Tt^2 + T^3) - a(T^2 - 2t \cdot t_{max} - 2t_{max}^2)t(t-T) = -at(t-T)(t^2 - Tt - 2t_{max}^2 + 2T \cdot t_{max})$.

Recall that at this point with running we assumed that the impulse, I^*, was known and that assumption led to the same polynomial approximation as assuming the maximum was known. Therefore for walking, we will strictly make assumptions about the function $F_W(t)$ to avoid a similar redundancy. Assuming $F_W(t_{max}) = M$, we find

$$F_W^{(1)}(t) = -\frac{t\left(T^2 t - 2\, t_{max}\, T^2 + 2\, T {t_{max}}^2 - 2\, t^2 T + 2\, T t t_{max} + t^3 - 2\, t {t_{max}}^2\right) M}{\left(-t_{max} + T\right)^2 {t_{max}}^2}$$

which is shown by the dashed curve in Figure 3.

Another quartic approximation with $F_W^{(2)}(0) = 0, F_W^{(2)}(T) = 0, F_W^{\prime(2)}(T/2) = 0, F_W^{(2)}(t_{max}) = M$, and $F_W^{(2)}(T/2) = m$ is shown by the dotted curve in Figure 3. The sextic polynomial approximation with $F_W(0) = 0, F_W(T) = 0, F'_W(0) = 0, F'_W(T) = 0, F'_W(T/2) = 0, F_W(t_{max}) = M$, and $F_W(T/2) = m$ is shown by the thin solid curve in Figure 3.

Notice with walking that higher order polynomials visually do a better job of approximating the experimental data. Once again, we can calculate the impulse for the aforementioned approximations to be 560.86 N·s for $F_W^{(1)}(t)$, 518.0 N·s for $F_W^{(2)}(t)$, and 460.9 N·s for $F_W(t)$. Recall that the calculated impulse from the experimental data was 438 N·s.

In running, we were able to use the idea of the average value of the function to assess the validity of the model. However, during a walk there is always a time when both feet are on the ground, i.e. there is no aerial phase. Therefore, the concept of calculating the average value doesn't apply to the walking gait.

Run with it

Below are some ideas for classroom and/or research projects for students.

1. Are there other, more physically reasonable, ways to approximate the VGRF curve for walking?

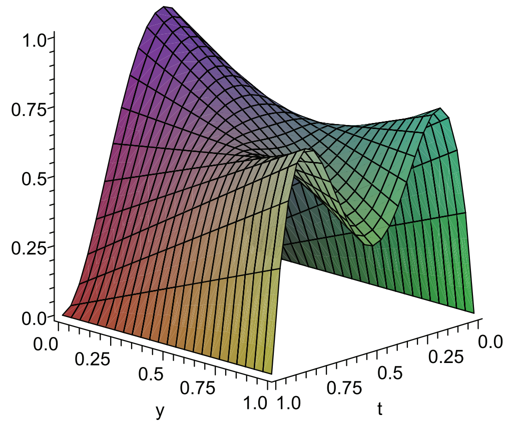

Figure 4: Surface to approximate the transition from walking to running. This surface uses the function f(t,y)=$F_R(t) \cdot (1-y) + F_W(t) \cdot y$ where we used a quartic approximation for both $F_R(t)$ and $F_W(t)$.

2. The above modeling assumes the two peaks in the VGRF curves are of equal height; is there a way to incorporate a curve having two different peak heights into the mathematical model?

3. In the above mathematical modeling, we have used polynomial interpolation, i.e. trying to use a polynomial to model the data. Is it possible, and beneficial, to use Fourier series or another set of basis functions to model the data?

4. A very interesting question is: what is happening in the transition between walking and running. We may conjecture that speed-walking is that transition, but why? In [3], a three dimensional surface is made from experimental data to interpolate what occurs during the transition from walking to running. It is straightforward to reproduce a modification of the surface in their paper, shown in Figure 4, given the models above. However, it doesn't make sense to have the impact time be the same in these very different gaits (recall how different the contact times are in Figure 1). What modifications can be made to the surface to make it more physically useful? Can the surface address any questions about speed-walking?

Conclusions

Sports biomechanics is replete with mathematical questions aimed at enhancing performance while simultaneously reducing the risk of injury. Injury usually occurs because of an overload to the musculoskeletal system. It is essential that we understand the forces involved in producing athletic movements as well as the features of these forces and movements that are associated with injury.

In this paper we have provided an introduction to some of the terminology and concepts of gait analysis using vertical ground reaction force in biomechanics. Mathematically we have only scratched the surface by using polynomials to model the VGRF curves that are seen in experimental data. We have also presented some applications of integration to the analysis of human gait, as well as some experimental data that can be used for educational purposes.

Acknowledgements

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